COMPUTER-AIDED METHODS FOR DEVELOPMENT OF SURFACES

by

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TO

MY

M A D H A V I

CERTIFICATE

This is to certify that the work entitled, " Computer-Aided Methods for Development of Surfaces" by P.B. Ramulu has been carried out under my supervision and has not been submitted elsewhere for a degree.

Dhande'

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P.B. Ramulu

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NOMENCLATURE

$A_{i,f}$	Area of the encasing rectangle defined by i^{th} and f^{th} vertices of a arbitrary polygon
A^*	Area of the optimal (minimum-area) rectangle
C_o	Corners of old encasing rectangle (a two dimensional array)
C_m	Corners of new encasing rectangle with area less than that of old one
$e_{i,i+j}$	Edge defined by i^{th} and $(i+j)^{th}$ vertices of the given arbitrary polygon
k	Curvature
$k(s)$	Curvature of a curve at arclength = s
k_g	Geodesic curvature
l	Slant height of a cone
P_i, Q_i	Starting points on primary and secondary curves respectively
P_f, Q_f	Last points on primary and secondary curves respectively
r	Radius of a cone
s	Arclength
$\dot{s}$	Derivative of arclength with respect to the parameter u

u, u', v	Parameters of parametric curves or a surface
$(u_i - u_f)$	Range of u
θ, β	Angles made by consecutive generators on the developed surface with the reference X-axis
α	Angle between the consecutive generators
σ_1	Primary curve
σ_2	Secondary curve
$\Delta\theta$	Angle of arc.

ABSTRACT

In the present work two methods of obtaining the development view of a surface have been proposed. The surface to be developed should either be a plane surface or a single-curved continuous surface or a composite of planar and single-curved continuous surfaces. The first method is based on the conventional triangulation method of development and it has been found to be suitable for the development of transition pieces. The second method is based on the principle that the geodesic curvature of a curve lying on a surface is the same as the curvature of the corresponding curve on the developed surface. This analytical method is found to be suitable for single-curved continuous surfaces. Also developed in the present work is an algorithm for obtaining the encasing rectangle of a polygon representing the development view of a surface such that the area of the encasing rectangle is minimum. Three computer programs have been developed and tested using several illustrative examples.

CHAPTER-1

INTRODUCTION

1.1 Development of Surfaces

In mechanical design practices it is often necessary to specify the geometry of surfaces. For instance the streamlined body-shapes of automotive vehicles, aircrafts, turbine blades, impellers etc.are illustrative examples in which surface design is an important aspect. Similarly several types of ducting systems are used in air-conditioning plants, furnaces and turbines. Designing of the shapes of these ducts is essentially a surface design problem.

Usually one has not only to design the surfaces but also to find the developed area of the specified surface. This is obtained by the process of development. Development is the process of laying out or unfolding a surface into a plane [6]. This is particularly necessary when the surface is built up of a metal sheet by folding appropriately without producing deformations. In such cases it is necessary to ensure that the designed surface is either fully developable or it consists of constituent surfaces each one of

which is developable. For instance a frustum of a cone is fully developable surface. On the other hand a transition piece connecting a circular boundary as one face to a rectangular boundary as the other is a piece-wise developable surface (See Figure 1.1).

Surfaces are classified as ruled surfaces and double curved surfaces [6]. In general a surface is considered to be the locus of a curve called the generatrix, when moved along another curve called the directrix. In case of ruled surfaces the generatrix is a straight line. In case of double curved surfaces the generatrix is a curve. Double curved surfaces are non-developable and hence will not be considered further in the present work.

Ruled surfaces are further classified as plane, single-curved and warped surfaces. All two dimensional figures are plane surfaces. These are obviously developable. Single-curved surfaces are generated by moving a straight line generatrix such that any two consecutive positions of it are either parallel or intersecting. Cone, cylinder, convolute are examples of single-curved surfaces. These surfaces are developable. Warped surfaces are generated by moving a straight line generatrix such that any two consecutive positions of the generatrix are not coplanar. Examples

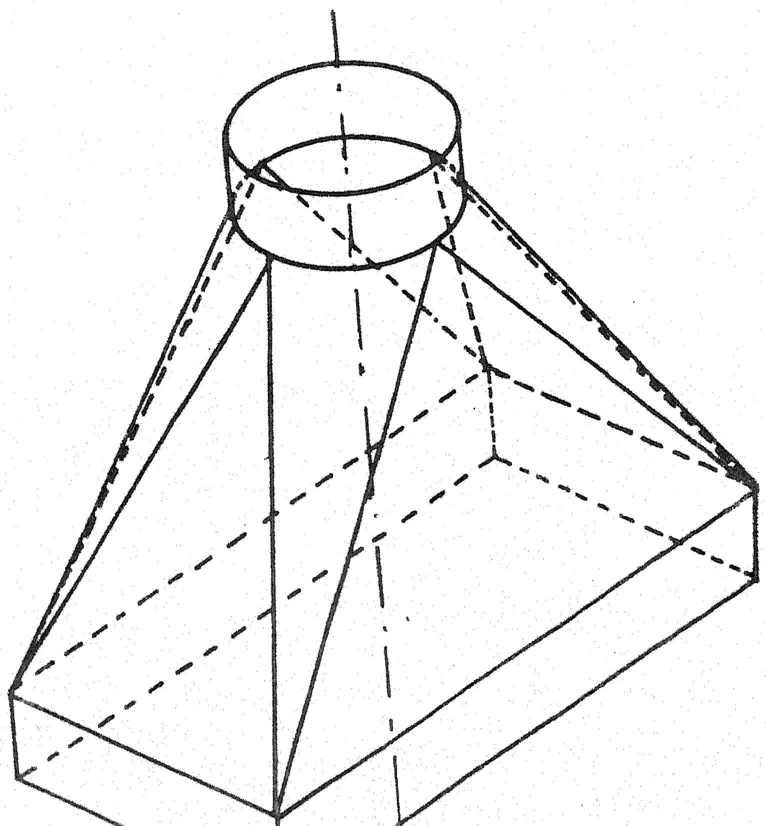


Fig. 1.1 A Transition Piece

of warped surfaces are the conoid, cylindroid, helicoid, hyperboloid, hyperbolic paraboloid [10]. Warped surfaces are non-developable.

Designers usually follow three general methods of development [6,7,8,9]. The Parallel Line Method is used for developing surfaces of prisms and cylinders. In this method the successive positions of the generatrix form a set of parallel straight lines. The Radial Line Method is used for developing surfaces of pyramids and cones. The successive positions of the generatrix in this method form a set of intersecting straight lines. (See Figures 1.2a and 1.2b)

The Triangulation Method is used for developing surfaces of transition pieces and other non-uniform connecting surfaces.

1.2 The Triangulation Method of Development

This is the method used in the approximate development of transition pieces as shown in Figures 1.3 and 1.4. The transition piece is considered to be composed of several constituent pieces or sections. Each of these pieces is ensured to be either a plane surface or a single-curved surface. The development of the given transition piece is obtained by first developing all constituent pieces and then concatenating

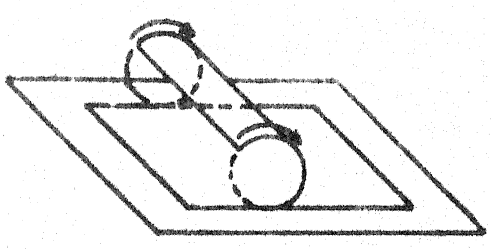


Fig. I. 2a Parallel Line Method

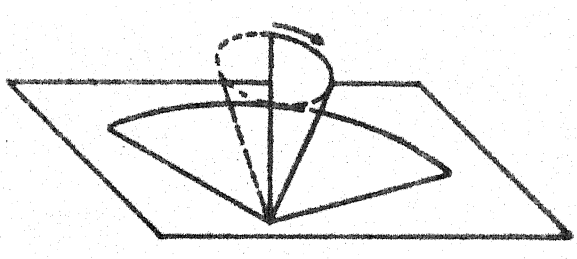


Fig. I. 2b Radial Line Method

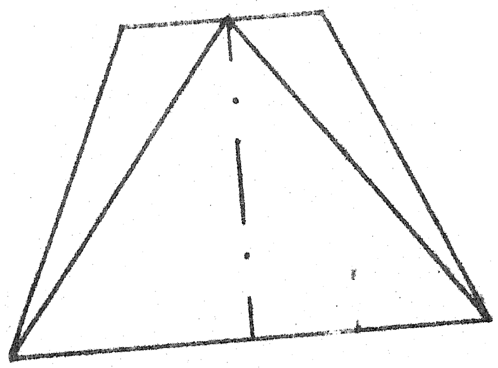


Fig. 1.3a Front View

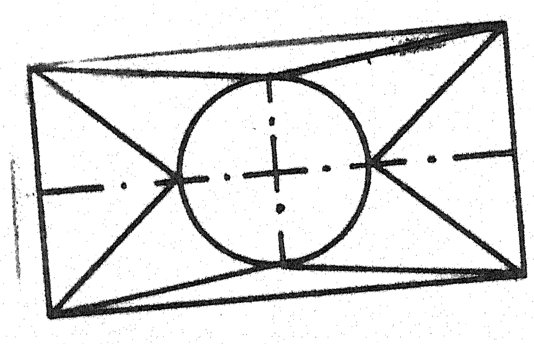


Fig. 1.3b Top View

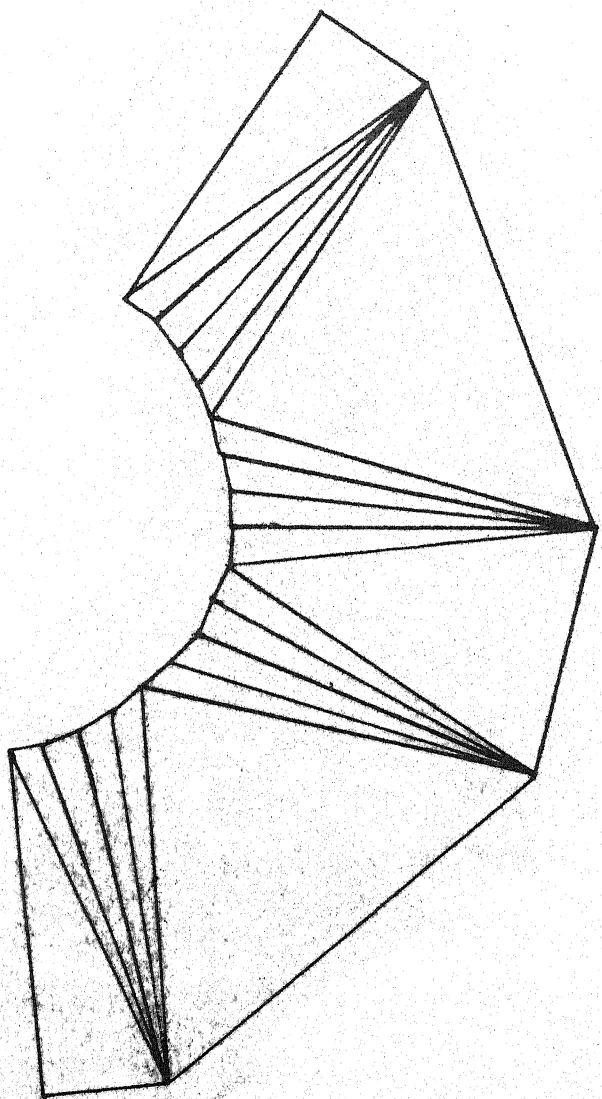


Fig. I.4 Development

developed areas of constituent pieces. The development of single-curved surfaces is obtained by considering it to be a composite of a large number of triangles. This approximation results in getting the approximate development of the given surface. (See Figure 1.4)

For the transition piece shown in Figure 1.3, one can specify a total of eight constituent pieces. Out of these, four are plane triangular surfaces and the remaining four are single-curved surfaces. The single-curved surfaces can be considered to be parts of oblique conical surfaces. The development of these single-curved surfaces is obtained by approximating each one of them by four triangular pieces as shown in Figure 1.4. The development of the total transition piece is also shown in this figure.

1.3 Literature Survey

The different methods describing how to get the development of single curved surfaces, transition pieces etc. have been described primarily in literature on descriptive geometry and engineering graphics. Several excellent books such as those of French and Vierck [8], Luzaddar [7], Street [6], Hawk [9] describe the graphical procedures for getting the development of a surface. As mentioned in Section 1.1 and 1.2,

all these methods are based on either the parallel line method or the radial line method or the method of triangulation. Though these methods are excellent for graphical procedures these are not convenient for mathematical modelling suitable for computerisation.

A mathematical approach describing how the geometry of a developed surface can be obtained from that of a curved surface is briefly indicated by Faux and Pratt. This approach is based on the fact that the curvature of a curve on the developed surface is equal to the curvature of the projection of the original curve on to the tangent plane, known as the Geodesic Curvature. The present work is based on both the method of triangulation as well as the mathematical definition of the geodesic curvature.

In order to display the curved surface as well as the developed surface in a graphical form on a computer graphics terminal one needs to know about several different techniques of geometric modelling and display of objects using Computer Graphics. For instance it is necessary to know as to what type of transformation matrices are to be used for displaying an isometric view of a transition piece. Literature on the methods of display techniques using Computer Graphics have been described by Roger and Adams, Faux and Pratt and Newman and Sproull[4,3,2].

1.4 Scope of the Present Work

In the present work two methods of development of developable surfaces are dealt, first being the analytical method and the second being the triangulation method. The analytical method including the related analytical geometry and the algorithm are described in Chapter 2. Similarly the triangulation method of surface development is described in Chapter 3. An algorithm to find the minimum-area rectangle which encases a given arbitrary polygon is described in Chapter 4. Computational results and conclusions are given in Chapters 5 and 6, respectively.

CHAPTER-2

ANALYTICAL METHOD OF SURFACE DEVELOPMENT

2.1 Introduction

A surface is generated by a family of curves. In parametric form a surface can be defined by the vector equation

$$\underline{r} = \underline{r}\ (u,\ v) \qquad (2.1)$$

When $u = u_i$ and v varies from v_{min} to v_{max} a constant-u curve is obtained. Similarly keeping $v = v_i$ and varying u from u_{min} to u_{max} one can get a constant-v curve. As shown in Figure 2.1a constant-u and constant-v parametric families of curves form a patch of a surface. If both families of curves have non-zero curvature then the surface generated is a double curved surface. For example, sphere is a double curved surface with contant-$\emptyset$ and constant-θ circles as shown in Figure 2.1b. If one family is a family of straight lines and the other, a family of non-zero curvature curves then the surface generated is a Ruled Surface. Cylinder is a ruled surface with constant-z circles and constant-θ straight lines, as shown in Figure 2.1c.

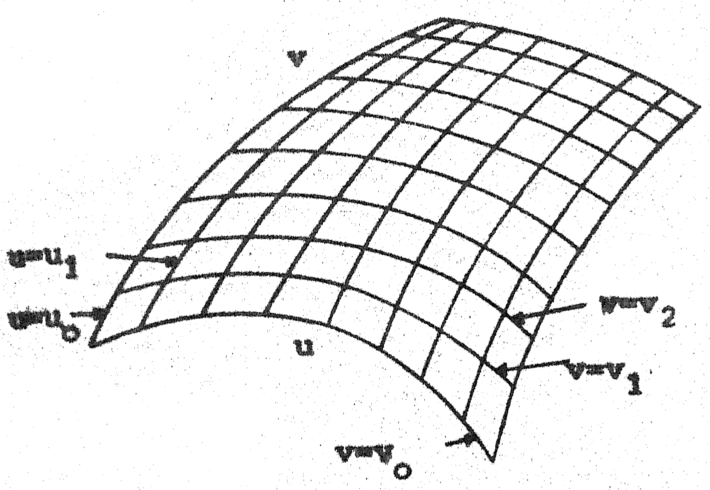


Fig. 2.1a

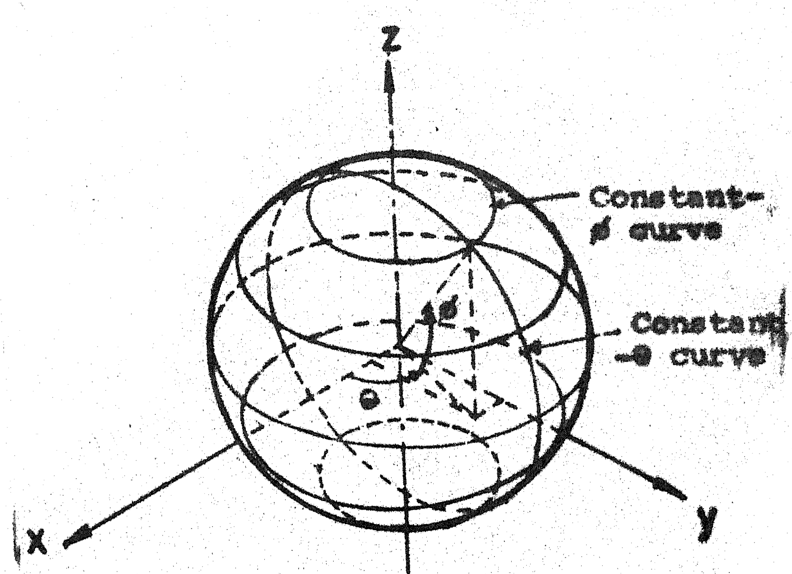


Fig. 2.1b

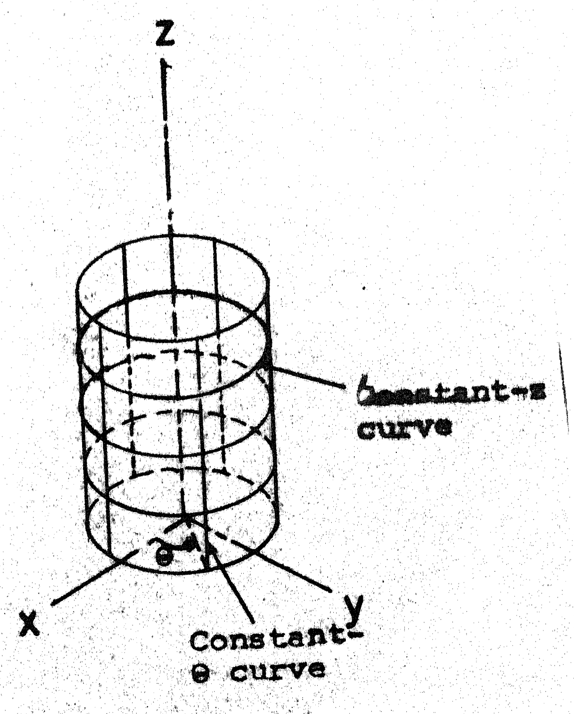


Fig. 2.1c

2.2 Developability

A ruled surface as shown in Figure 2.2a can be expressed in general by the equation

$$\underline{r} = \underline{r}\ (u,\ v) = \underline{r}_0(u) + v\ \underline{m}\ (u) \qquad (2.2)$$

where $\underline{r}_0(u)$ is a given point on the line whose parameter is u and $\underline{m}$ (u) is a unit direction vector of the orientation of the generator. It can be seen that the parameter v gives the distance of the point $\underline{r}$ (u, v) from $\underline{r}_0(u)$. An alternate expression based on rulings joining corresponding points on two space curves $\underline{r} = r_1(u)$ and $\underline{r} = r_2$ (u) is given by

$$\underline{r} = \underline{r}\ (u,\ v) = (1 - v)\ \underline{r}_1\ (u) + v\ \underline{r}_2\ (u) \quad \ldots. \quad (2.3)$$

In both cases $\underline{r} = \underline{r}_0$ (u) as well as $\underline{r} = \underline{r}_1$ (u) and $\underline{r} = \underline{r}_2$ (u) are known as directrices and the rulings are called generators or generatrices.

A surface is developable if it can be unrolled into a plane without distortion. Consider the ruled surface expressed by the Equation (2.3). It is proved [3] that the surface is developable if

$$(\ \underline{r}_1 - \underline{r}_2\ ) \cdot (\ \dot{\underline{r}}_1 \times \dot{\underline{r}}_2\ ) = 0 \qquad (2.4)$$

where

$$\dot{\underline{r}}_1 = \frac{d\underline{r}_1}{du} \quad \text{and} \quad \dot{\underline{r}}_2 = \frac{d\underline{r}_2}{du}$$

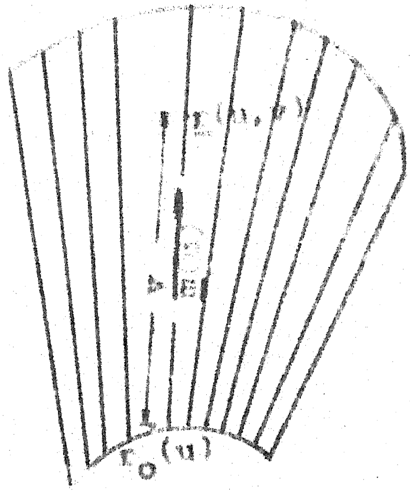


Fig. 2.2a

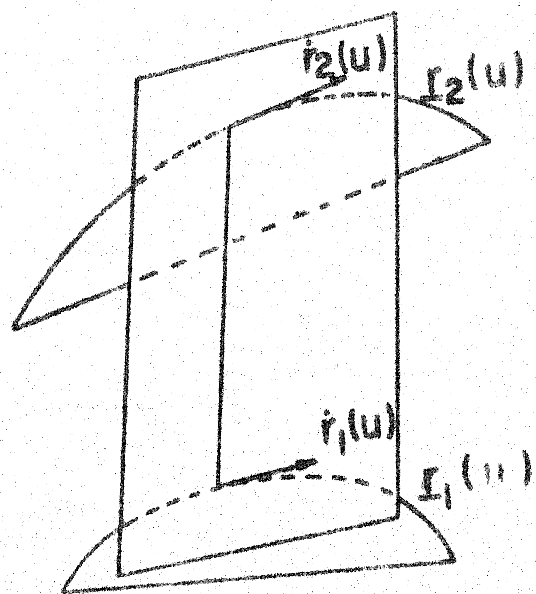


Fig. 2.2b

The equation (2.4) implies that the tangents $\dot{\underline{r}}_1$ and $\dot{\underline{r}}_2$ and the generator ($\underline{r}_1 - \underline{r}_2$) should be coplanar.

2.3 Tangent Plane Generation of a Developable Surface

Considering the parametrisation of $\underline{r}_1(u)$ to be known, one may choose the parametrisation of the second curve in such a way that the condition given in Equation 2.4 is satisfied. As shown in Figure 2.2b the parameter u is assigned to the point on $\underline{r}_1$ at which a plane tangent to $\underline{r}_1$ at parameter value u is tangent to $\underline{r}_2$ and this is the basis of the tangent plane generation of a developable surface containing two given curves.

The development is based on the fact that the curvature of a curve on the developed surface is equal to the curvature of the projection of the original curve onto the tangent plane, known as the Geodesic Curvature, k_g. (See Figure 2.3). The geodesic curvature of any curve $\underline{r} = \underline{r}\ (u)$ is given by [3]

$$k_g = \frac{\underline{n} \cdot (\ \dot{\underline{r}} \times \ddot{\underline{r}}\ )}{\dot{s}^3} \qquad (2.5)$$

where

$\underline{n}$ - is the unit normal to the surface,

$\dot{s}$ - $= \left| \frac{d\underline{r}(u)}{du} \right|$ and

$\ddot{\underline{r}}$ - $= \frac{d^2\underline{r}(u)}{du^2}$

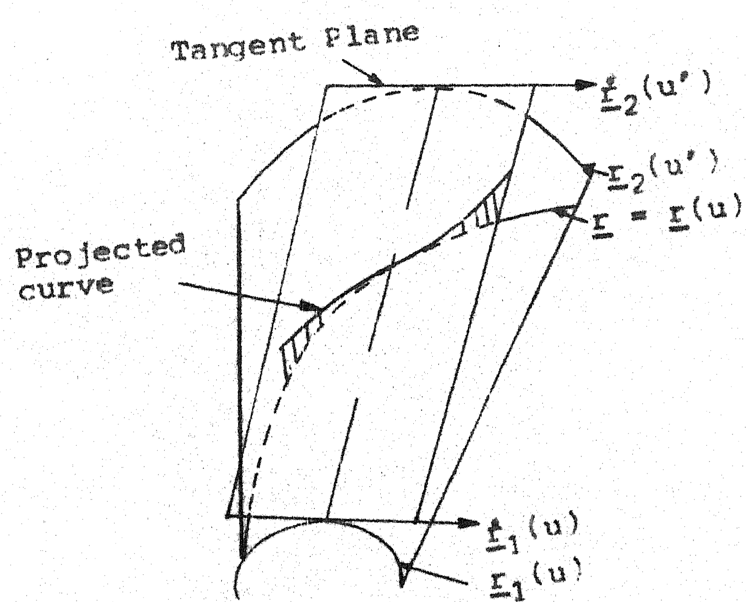
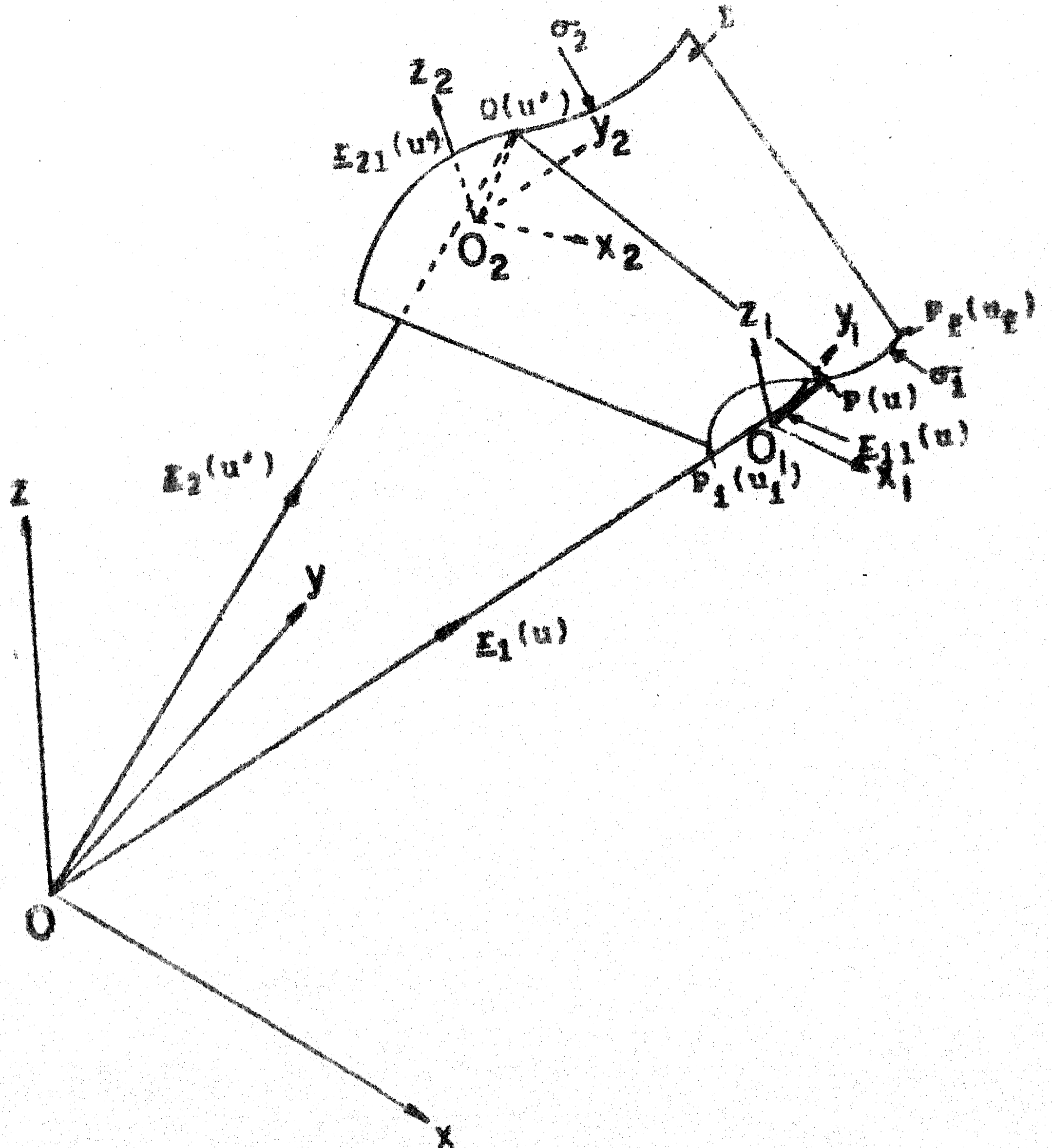

Tangent Plane
$\underline{t}_2(u')$
$\underline{r}_2(u')$
$\underline{r} = \underline{r}(u)$
Projected curve
$\underline{t}_1(u)$
$\underline{r}_1(u)$

Fig. 2.3

As an example, consider a right circular cone of radius r and slant height l. Its base and the vertex are the directrices. The development of the cone is a sector of a circle. The geodesic curvature of the base circle of radius r is the curvature of the circle of which the sector is a part, which, for the right circular cone, is the reciprocal of the slant height (= 1/l).

As shown in Figure 2.4, let σ_1 and σ_2 be the given directrices. The radius vector of a generic point lying on the directrix σ_1 is defined with respect to a local coordinate system S_1 (O_1 - X_1 Y_1 Z_1) as $\underline{r} = \underline{r}_{11}(u)$. Similarly the directrix σ_2 is defined with respect to another local coordinate system S_2 (O_2 - X_2 Y_2 Z_2) as $\underline{r} = r_{21}(u')$ where u and u' are the parameters. Let $[M_{01}]$ be the transformation matrix between the local coordinate system S_1 (O_1 - X_1 Y_1 Z_1) and the global coordinate system S (O - X Y Z). Similarly $[M_{02}]$ is the transformation matrix between S_2 (O_2 - X_2 Y_2 Z_2) and S (O - X Y Z). The equations of the curves σ_1 and σ_2 in the global coordinate system are,

$$\left.\begin{aligned} \underline{r}_1(u) &= \underline{r}_{11}(u)\ [M_{01}] \\ \underline{r}_2(u') &= \underline{r}_{21}(u')\ [M_{02}] \end{aligned}\right] \qquad (2.6)$$



σ2
Z2
Q(u')
y2
r21(u')
O2
x2
Z1
y1
pf(uf)
σ1
p(u)
r11(u)
O1
x1
p1(u1)
r2(u')
Z
y
r1(u)
O
x

Fig. 2.4

where

$$\underline{r}_1(u) = [\ x_1(u) \quad y_1(u) \quad z_1(u) \quad 1\ ]$$

$$\underline{r}_2(u') = [\ x_2(u') \quad y_2(u') \quad z_2(u') \quad 1\ ]$$

If the given primary curve is $\underline{r} = \underline{r}_1(u)$ and secondary curve is $\underline{r} = \underline{r}_2(u')$ as shown in Figures 2.3 and 2.4, the developability condition using Equation (2.4) is as follows:

$$[\ \underline{r}_1(u) - \underline{r}_2(u')\ ] \cdot \dot{\underline{r}}_1(u) \times \dot{\underline{r}}_2(u') = 0 \qquad \ldots \qquad (2.7)$$

The developable surface which satisfies Equation 2.7 is

$$\underline{r} = \underline{r}\ (\ u,\ v\ ) = (\ 1 - v\ )\ \underline{r}_1(u) + v\ \underline{r}_2(u') \qquad \ldots \qquad (2.8)$$

For a given value of u for the primary curve one has to solve Equation 2.7 for u′ so that the surface is developable. In the present work Newton - Raphson method is used to solve Equation (2.7). One can then calculate the outward normal to the surface, $\underline{n}$ by the following equation.

$$\underline{n} = \frac{\dfrac{\partial \underline{r}(u,v)}{\partial u} \times \dfrac{\partial \underline{r}(u,v)}{\partial v}}{\left| \dfrac{\partial \underline{r}(u,v)}{\partial u} \times \dfrac{\partial \underline{r}(u,v)}{\partial v} \right|} \qquad (2.9)$$

Substituting Equation (2.8) in (2.9) and simplifying, one can get

$$\underline{n} = \frac{\frac{d\underline{r}_1}{du} \times (\underline{r}_2(u') - \underline{r}_1(u))}{\left| \frac{d\underline{r}_1}{du} \times (\underline{r}_2(u') - r_1(u)) \right|} \tag{2.10}$$

The equation for the geodesic curvature for the primary curve σ_1 , obtained by using Equation (2.5), is as follows:

$$k_g = \frac{\underline{n} \cdot (\frac{dr_1}{du} \times \frac{d^2}{du^2} (\underline{r}_1(u)))}{\dot{s}^3} \tag{2.11}$$

where $\underline{n}$ - is the unit outward normal to the surface

$\dot{s}$ - $= \left| \frac{d\underline{r}_1(u)}{du} \right|$

After finding the curvature of the primary curve using Equation (2.11), one may obtain the developed curve itself by integrating the Serret - Frenet Equation [3]

$$\begin{aligned} \frac{d^2x}{ds^2} + k(s) \frac{dy}{ds} &= 0 \\ \frac{d^2y}{ds^2} - k(s) \frac{dx}{ds} &= 0 \end{aligned} \tag{2.12}$$

where k(s) is the curvature which happens to be the geodesic curvature. The numerical scheme of integration by fourth order Runge-Kutta method is explained in Appendix I. As the length of the generator can be calculated by knowing u on $\underline{r}_1$ and corresponding u' on

$\underline{r}_2$ one may obtain the development of point Q by noting that the angle between the tangent of the primary curve ($\underline{r}_1$) and the generator is unchanged during development.

2.4 The Algorithm

Step 1. Divide the range of u of the primary curve, u_i to u_f , in m equal parts. Let P_i , $P_{i+1}, \ldots\ldots, P_{i+j}, \ldots\ldots, P_{i+m} = P_f$ be the points on the (primary) curve corresponding to the values u_i, $u_{i+1}, \ldots . u_{i+j}, \ldots ., u_{i+m}$ respectively.

Step 2. Compute for each P_{i+j} ($j = 0, 1, \ldots .,m$) the corresponding points Q_{i+j} on the secondary curve by solving Equation (2.7) for the parameter u′ at each P_{i+j} .

Step.3. For each generator P_{i+j} ($j = 0, 1, \ldots .,m$), find the unit normal vector $\underline{n}_{i+j}$ using Equation (2.10).

Step 4. For each P_{i+j} ($j = 0, 1, \ldots ., m$), find the geodesic curvature $k_{g(i+j)}$ using Equation (2.11).

Step 5. For the first point P_i, assume $s_i = 0$. For the successive points P_{i+1} , $P_{i+2}, \ldots,$ $P_{i+m} = P_f$, find the values of arclengths

$s_{i+1}, s_{i+2}, \ldots, s_{i+m} = s_f$ using the equation

$$s_{i+j+1} = s_{i+j} + \left[\frac{d\underline{r}_1(u)}{du} \right]_{u_{i+j} + \frac{\Delta u}{2}} \cdot (u_{i+j+1} - u_{i+j}) \qquad j = 0, 1, 2, \ldots, (m-1)$$

Step 6. Plot a graph of $k_{g(i+j)}$ versus s_{i+j} on k-s plane. It is assumed that variation between any two consecutive values of k_g is linear as shown in Figure 2.5.

Step 7. Using the information of the k-s graph of step 6 and the initial conditions that at P_i

$$s = s_i = 0,\ k = k_i\ ;\ x = 0\ ;\quad y = 0\ ;$$

$$\frac{dx}{ds} = 1.0\ ;\quad \frac{dy}{ds} = 00,$$

integrate the Serret-Frenet equations given in Equation (2.12). Note that the development curve x-y of the primary curve σ_1 has been obtained in a tangent plane at P_i with a two dimensional coordinate system whose origin is at P_i and the X-axis is aligning with the tangent $\frac{d\underline{r}_i}{du}$ at P_i.

Step 8. Once the set of values of (x_{i+j}, y_{i+j}) $(j = 0, 1, \ldots, m)$ for the primary curve are obtained, the corresponding points for the

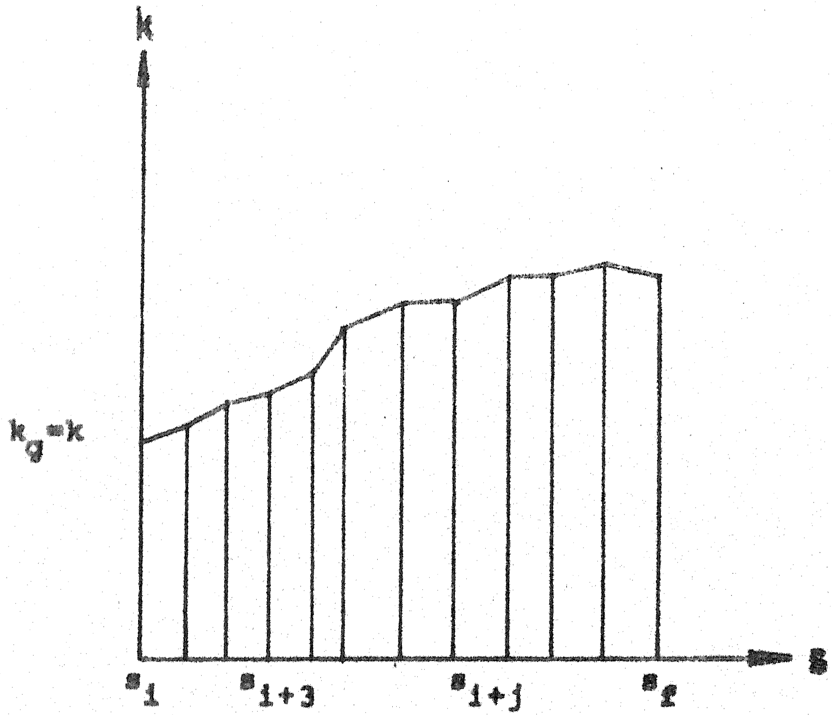


Fig. 2.5 Primary curve profile

secondary curve can be obtained by knowing that the angle between the tangent of the primary curve and the generator does not change during development.

CHAPTER-3

TRIANGULATION METHOD OF SURFACE DEVELOPMENT

3.1 Graphical Method

The triangulation method is one of the three graphical methods (illustrated in Section 1.1) used for developing surfaces. This method is used for developing surfaces of transition pieces and other non-uniform surfaces which are used to connect ducts, pipes and similar objects with various different sizes and shapes.

As an example of (triangulation method by) graphical method, consider the development of a transition piece shown in Figure 3.1. The geometry of the transition piece is specified by giving its top view and front view. The circular base is divided into sixteen equal arcs, and the division points are connected to corners of the rectangular top by straight lines, also referred to as element lines. Since the arclengths ab, bc etc. can not be measured graphically, these are approximated by their chordlengths and thus each of the four parts designated as B, D, F and H is divided into four triangles. This divides the transition piece into sixteen small triangles and four large triangles corresponding to the parts A, C, E and G. To reduce

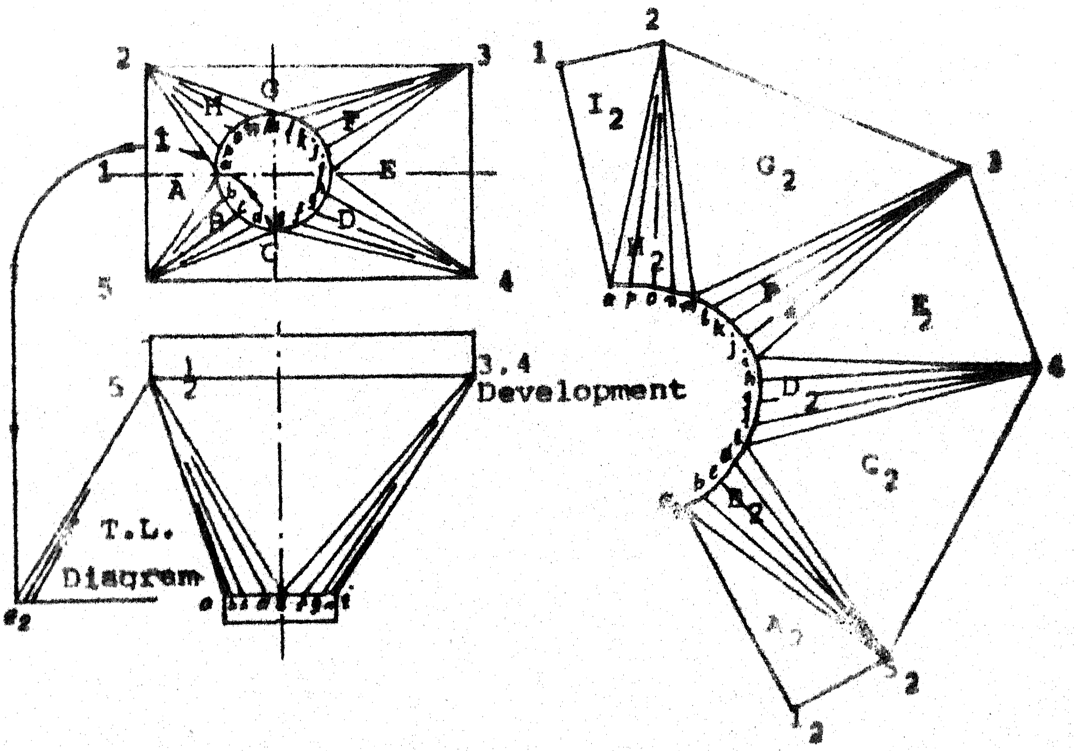


Fig. 3.1 Graphical Method of Triangulation

the weld-length, the transition piece is cut open along the shortest generator and then developed into a plane. It is the generator a-1 as shown in the Figure 3.1. True length diagrams are also shown in the figure. The true length of e5 is found at the left of the views by revolving the line parallel to frontal plane. The true length of a1 is positioned at $a_2 1_2$. Using a_2 as the centre and the true length of a5 as radius an arc is described at 5_2. With the true length of 1-5 as radius and 1_2 as the centre the point 5_2 is fixed and layout of triangle A is completed at A_2. Then triangles B_2, C_2, D_2, E_2, F_2, G_2, H_2 and I_2 in succession are completed to get the flat pattern.

3.2 Computer Graphics Method

The graphical method illustrated in Section 3.1 is computerised with a few modifications.

(i) The top and base of the transition piece are defined with respect to their local coordinate systems. The corresponding transformation matrices with respect to a global coordinate system are specified so that the geometry of the transition piece is defined with respect to the global coordinate system.

(ii) Instead of chordlength, the arclength is computed using the formula

$$\text{arclength} = R \, . \, \Delta\theta \qquad (3.1)$$

where R is the radius of the circle and $\Delta\theta$ is the angle of the arc. This improves the accuracy of the method.

(iii) The true lengths of generators are computed using the formula for distance between two given points. If P (x_1 , y_1 , z_1) and Q (x_2 , y_2 , z_2) are two given points then

$$\text{length } PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \quad \ldots \quad (3.2)$$

(iv) Laying out of large triangles is done as follows. As shown in Figure 3.2a, let ION be the large rectangle to be laid out. ION′ is the previously laid out triangle. Let θ be the angle made by IO with X-axis. The angle α is computed using the formula

$$\cos\alpha = \frac{IO^2 + NI^2 - ON^2}{(2 \, . \, IO \, . \, NI)} \qquad (3.3)$$

Lengths of IO, ON and NI are computed using Equation (3.2).

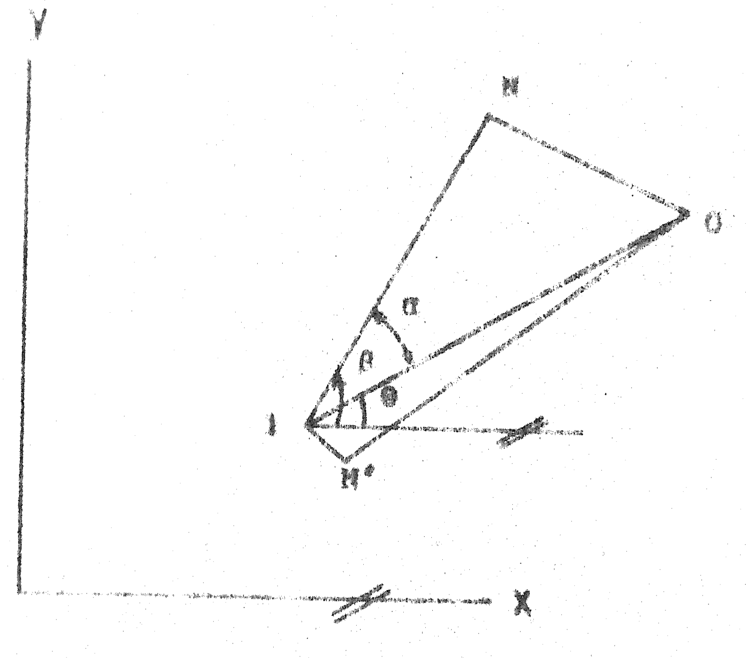


Fig. 3.2a

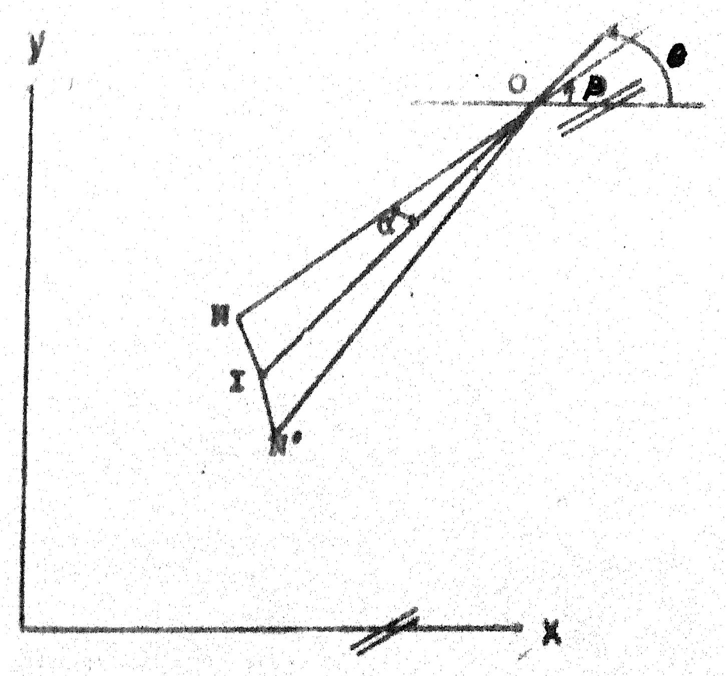


Fig. 3.2b

Let (x_i , y_i) be the coordinates of the point I. The coordinates (x_n, y_n) of point N are computed as follows.

$$x_n = x_i + NI.\cos\beta$$
$$y_n = y_i + NI.\sin\beta \qquad (3.4)$$

where $\beta = \alpha + \theta$.

In a similar way the small triangles are laid out, as follows.

As shown in Figure 3.2b, ION' is the previously laid out triangle. The side IO makes an angle θ with X-axis. It is required to find the coordinates of point N, knowing the lengths ON and NI. The angle α is calculated by the formule

$$\cos\alpha = \frac{IO^2 + ON^2 - NI^2}{(2 . IO . ON)} \qquad (3.5)$$

If (x_o , y_o) are the coordinates of point O then the coordinates of the point N (x_n, y_n) are computed as follows.

$$x_n = x_o - ON.\cos\beta$$
$$y_n = y_o - ON.\sin\beta \qquad (3.6)$$

where $\beta = \theta - \alpha$

CHAPTER-4

OPTIMAL ENCASING RECTANGLE

4.1 Introduction

The development of any surface, for instance, frustum of a cone, transition piece etc. is basically a polygon of many sides. The developed surface is sometimes known as the pattern. The pattern is generally cut out of a rectangular sheet of metal, wooden plank or a card borad. As shown in Figure 4.1, it can be seen there exist quite a few rectangles which encase a given polygon. To optimize waste, it is required to find that particular rectangle whose area is the minimum. In addition to that, it is necessary to specify the coordinates of all the vertices of the polygon with respect to a rectangular coordinate system whose origin may be any suitable corner of the optimal (minimum-area) rectangle and whose axes are the corresponding sides of the same rectangle.

4.2 The Algorithm

Let an arbitrary polygon be defined with respect to a rectangular coordinate system, as shown in Figure 4.2a. It is to be noted that the referred

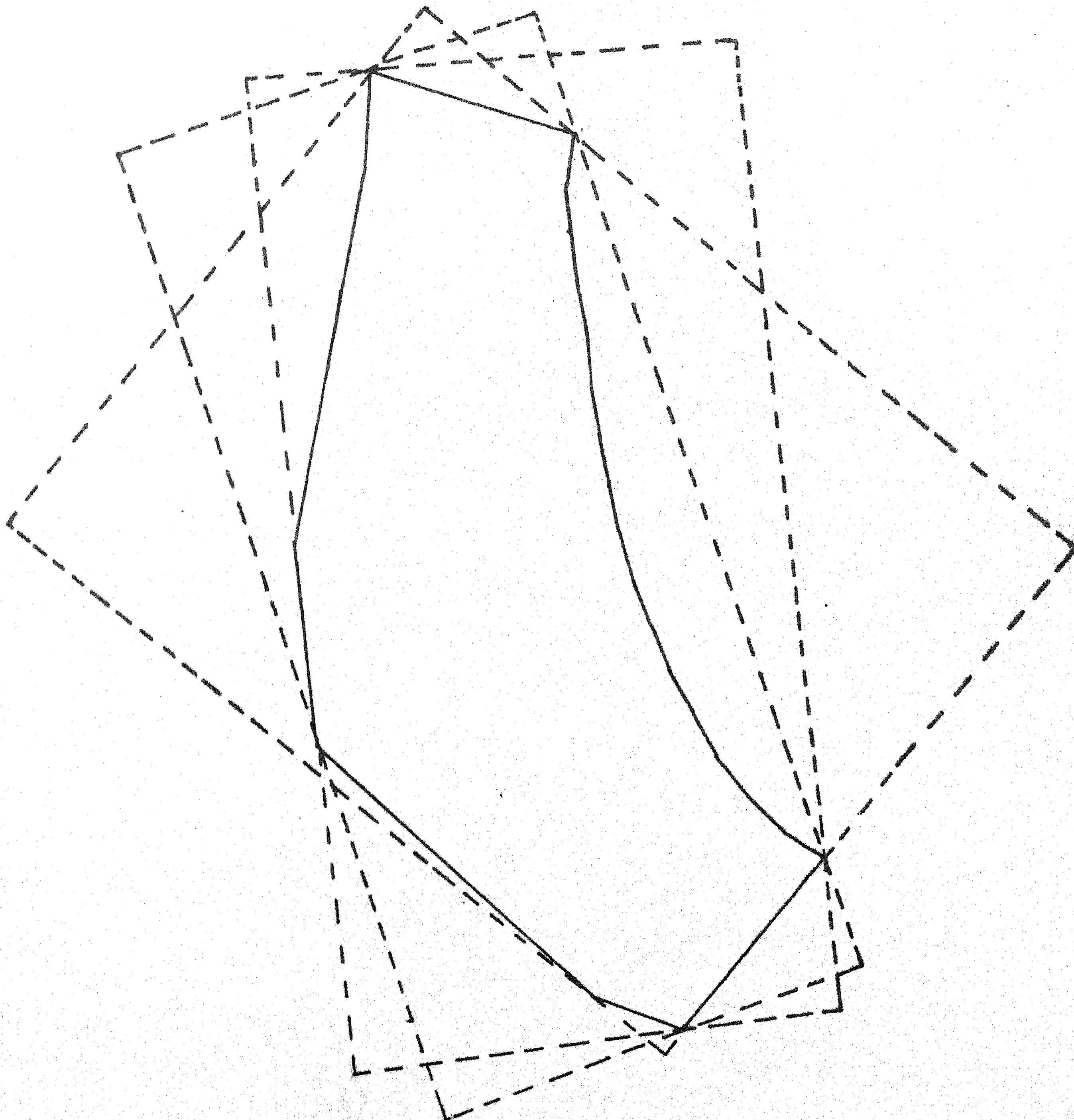


Fig. 4.1 Encasing rectangles

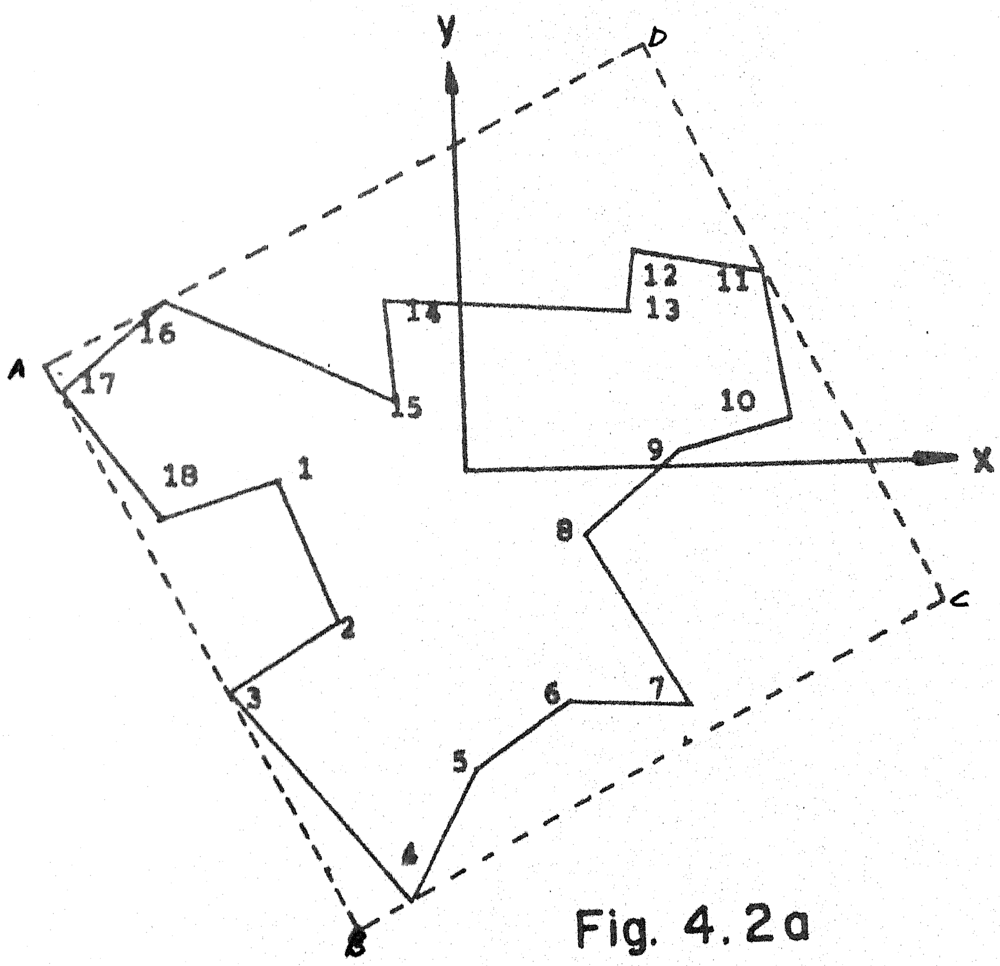


Fig. 4.2a

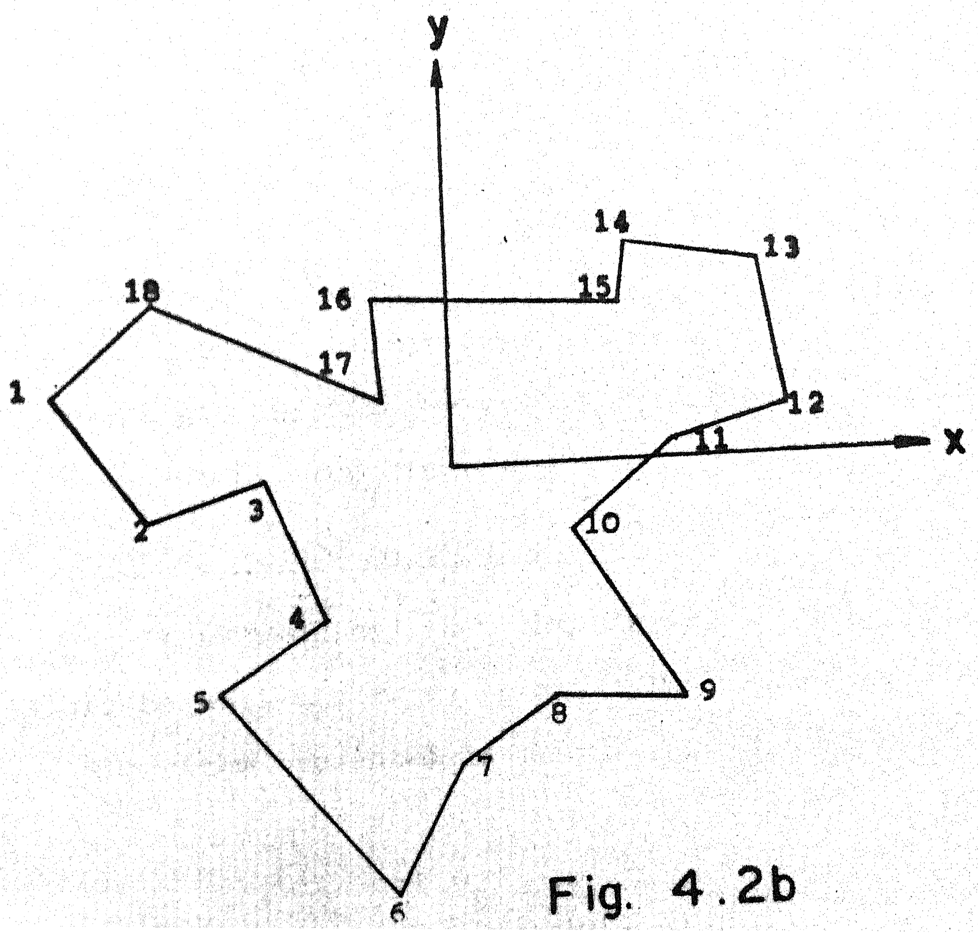


Fig. 4.2b

coordinate system is the same with respect to which the developed surface has been obtained using the procedures described in Chapters 2 and 3.

In principle, one can have an infinite number of solutions for obtaining a circumscribing rectangle to a developed surface-polygon. However, only a subset of this infinite number of solutions need to be considered for finding the optimum size rectangle. It can be observed from Figure 4.2a that all these rectangles which encase the given arbitrary polygon will have atleast one side passing through any two vertices of the polygon. In the Figure 4.2a, the side of the rectangle ABCD is AB and the vertices are 17 and 3. It is now required to find all such possible encasing rectangles, their areas, and then find, out of those rectangles that rectangle with minimum area.

4.2.1 Numbering of Vertices

To reduce computation, it is required to number the vertices in such a way that the first vertex has the minimum x coordinate.

4.2.2 Renumbering According to Direction

It is required that the numbering of vertices should be anti-clockwise so as to be consistent with the algorithm presented in this work. So if the

original numbering of vertices is clockwise, it is necessary that the vertices should be renumbered in anti-clockwise direction beginning with the x-minimum vertex as the first one. The polygon shown in Figure 4.2a after renumbering is shown in Figure 4.2b. Now one need to start finding the possible encasing rectangles.

Let 1, 2, 3,...., n be the number of vertices formed based on 4.2.1 and 4.2.2 with $x_1 = x_{min}$. Put an additional number (n + 1) on the first vertex 1.

Step 1: Begin with vertex $V_i = 1$.

Step 2: Test the edge $e_{i,i+1}$ formed by the next vertex V_{i+1}, that is the line joining V_i to V_{i+1} for its feasibility using the method explained in the Appendix II. If this line or any of its extended portions transpass over the area of the given polygon then skip the edge $e_{i,i+1}$ and proceed to the vertex V_{i+2}. Repeat the test on the edge $e_{i,i+2}$ formed by joining V_i to V_{i+2}. Continue till an edge passes the test, say $e_{i,i+j}$. Call that vertex as $V_f = V_{i+j}$.

Step 3: The edge $e_{i,i+j}$ defines one side of the encasing rectangle. Construct the circimscribing rectangle with $e_{i,i+j}$ as part of one side and

find out the area, $A_{i,f}$. The method for finding the corners and area of the rectangle is explained in Appendix-III. Store this data of the rectangle.

Step 4: If $V_f = V_{n+1}$, go to step 6, otherwise proceed to step 5.

Step 5: Set $V_i = V_f$ and proceed to step 2.

Step 6: Select the minimum from the list of values $A_{i,f}$. Let the minimum be at $i = i^*$ and $f = f^*$ So, the minimum area is encompassed by forming a rectangle with the vertices i^* and f^* lying on one of its sides. The minimum value is A^*.

Step 7: Take any one corner of the rectangle corresponding to the area A^* as the origin and the two sides meeting at the corner as the coordinate axes. Transform the original values of coordinates of all vertices to a new set of values with respect to the rectangle-corner coordinate system.

The algorithm explained above is represented in the form of a flow diagram in Figure 4.3.

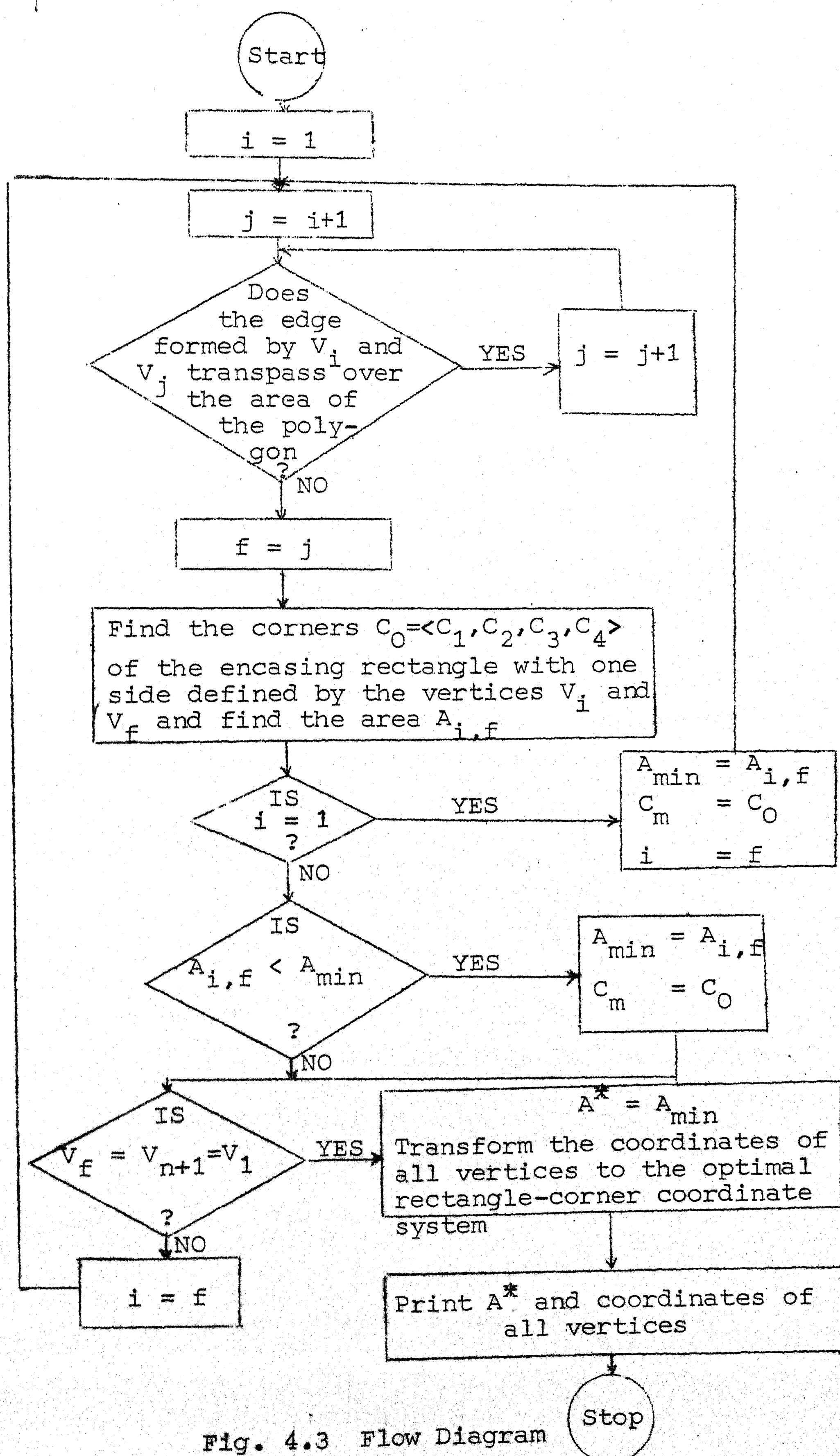


Fig. 4.3 Flow Diagram

CHAPTER-5

COMPUTATIONAL RESULTS

5.1 Computer Programs

Based on the analytical method of surface development described in Section 2.4, a computer program in PASCAL language has been developed. The program is general enough to develop any surface which satisfies the condition of developability. The surface can either be a single continuous one, as for example, a frustum of a cone, or it can be consisting of many pieces each one of which is a developable surface, such as a transition piece. The directrices can be points, straight lines, conics or Bezier curves.

For the triangulation method of surface development, three programs have been developed. The first one, programmed in PASCAL, is for developing the surface of a transition piece with two openings, one being circular and the other rectangular. The second program, written in FORTRAN-10, is for the development of a transition piece with one opening circular and the other one a closed polygon of any number of sides. The third

program, in FORTRAN-10, is for the development of a surface of a transition piece with conic-section openings. The algorithm for all the three programs is given in Section 3.2. When the programs for both analytical and triangulation methods are executed the output is obtained in two forms. First, the top view, the front view and the development in the form of drawings are displayed on the screen of a graphics terminal. If necessary, the plotter output can be obtained by the user. As a second step, the output may also be printed in the form of tables on a line-printer indicating the x-y coordinates of different points of the developed surface.

In order to find out the optimal encasing rectangle for a given arbitrary polygon, a computer program in FORTRAN-10, has been developed. The algorithm used for this purpose is given in Section 4.2. For a given polygon the program finds the optimal rectangle, transforms the coordinates to a suitable corner-coordinate system and gives the transformed development data in the form of line-printer output. This data is useful for actually cutting patterns to a true scale from a sheet of metal or cardboard. The program also gives the area of a given arbitrary polygon, the area of the encasing optimal rectangle and percentages of sheet metal or

cardboard used and unused. The program has been successfully tested with many examples.

5.2 Examples

For easy understanding of the definition of a developable surface, it is necessary to define primary and secondary curves with respect to their local coordinate systems and specify the corresponding transformations. Figure 5.1 shows local coordinate system, global coordinate system and the relation between them (rotation and translation). Though rotation about only Z axis is shown, one can have rotations about X and Y axes too.

5.2.1 Example 1

As an example for the analytical method of surface development a surface shown in Figure 5.2 is considered. The primary and secondary curves (circles in this example) are defined with respect to their local coordinate systems as shown in the figure. The input data for the example is shown in Table 5.1. The output is obtained in two forms. The development in the form of plotter output is shown in Figure 5.3. The development data, necessary for getting the development to true scale, in the form of line printer output giving x-y

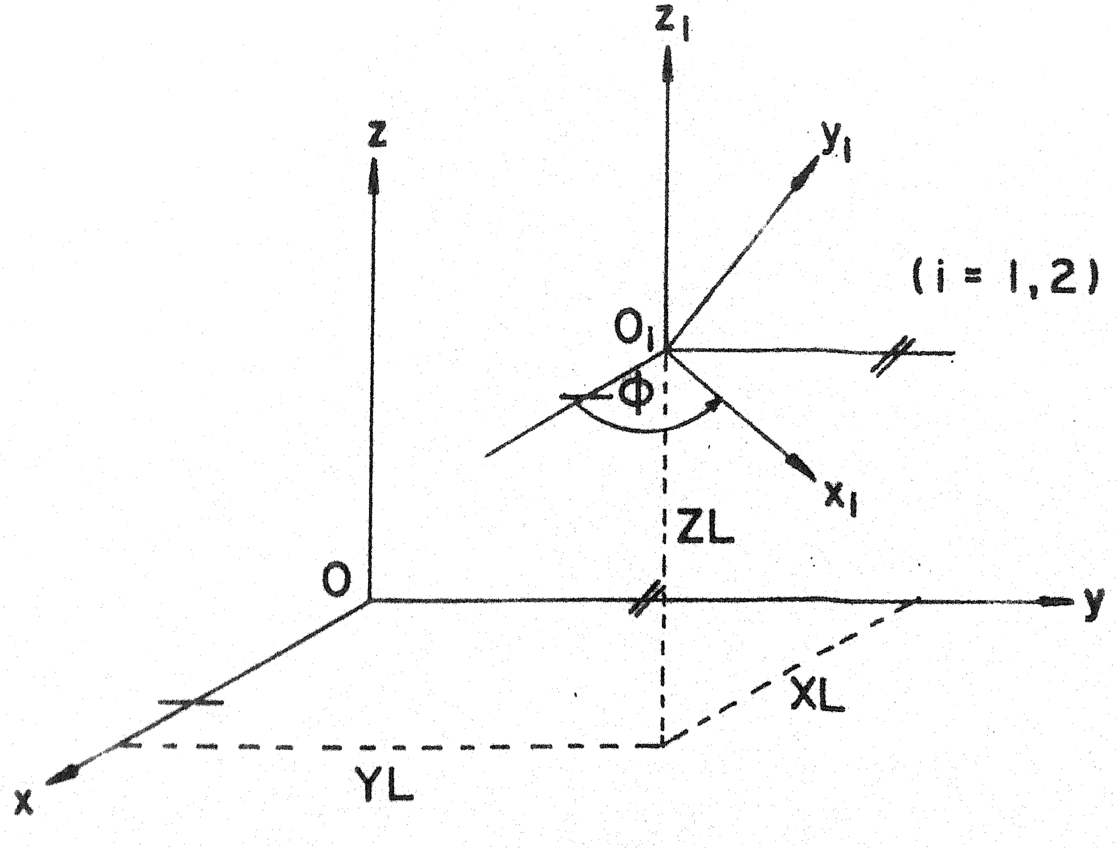
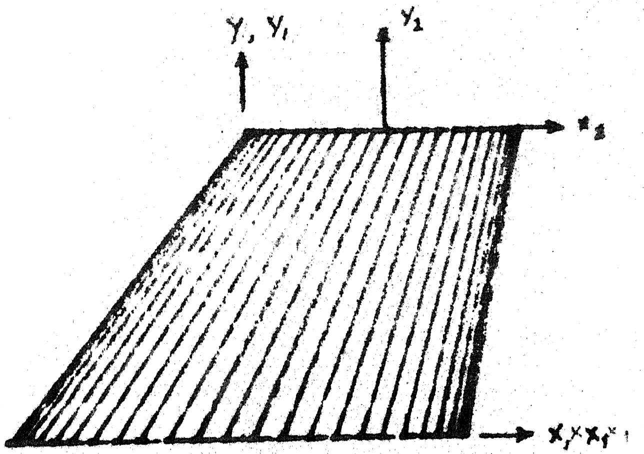


Fig. 5.1 Local and Global coordinate systems

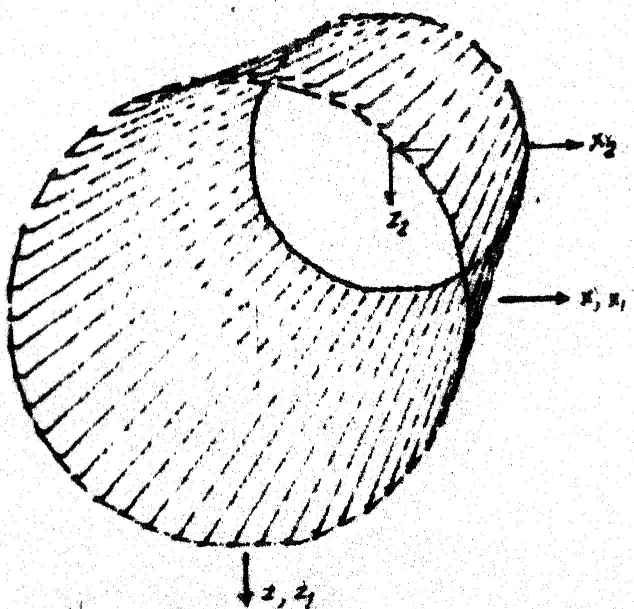
TABLE 5.1

Input data for Example 1

CURVE	CURVE NATURE	CURVE DEFINITION WITH RESPECT TO LOCAL COORDINATE SYSTEM.		TRANSFORMATIONS				NUMBER OF PARTS THE PRIMARY CURVE DIVIDED INTO
		MAJOR AXIS	MINOR AXIS	ROTATION ABOUT Z-AXIS	TRANSLATIONS			
					XL	YL	ZL	
PRIMARY	CIRCLE	3.0	3.0	0.0	0.0	0.0	0.0	50
SECONDARY	CIRCLE	1.8	1.8	0.0	2.0	4.0	-2.0	



Front View



Top view

Fig. 5.2

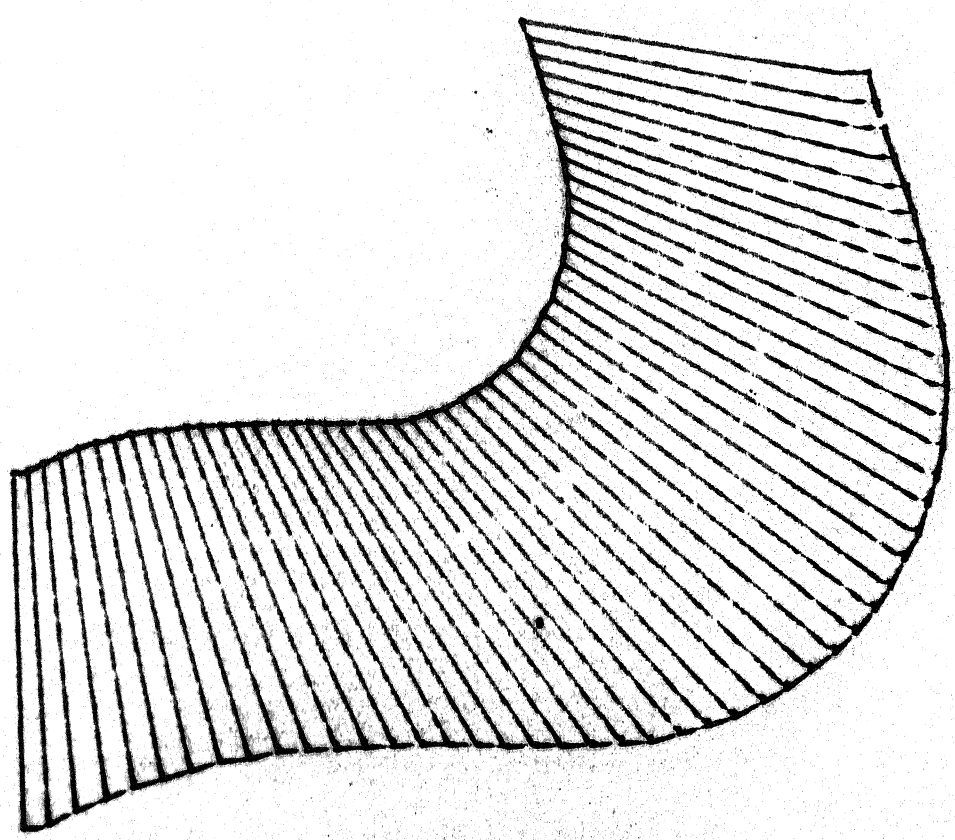


Fig. 5.3 Development

coordinates of different points on the developed primary and secondary curves can also be made available to the user upon exercising appropriate option.

5.2.2 Example 2

As an example for the modified triangulation method of surface development, a transition piece is considered. Figure 5.4 shows the top and front view of the transition piece. It can be seen that one opening is rectangular and the other is circular. The input data for the example is given in Table 5.2. The output is in two forms. The development in the form of plotter output is shown in Figure 5.5. The line-printer output showing x-y coordinates of different points on the developed surface is given in Table 5.3.

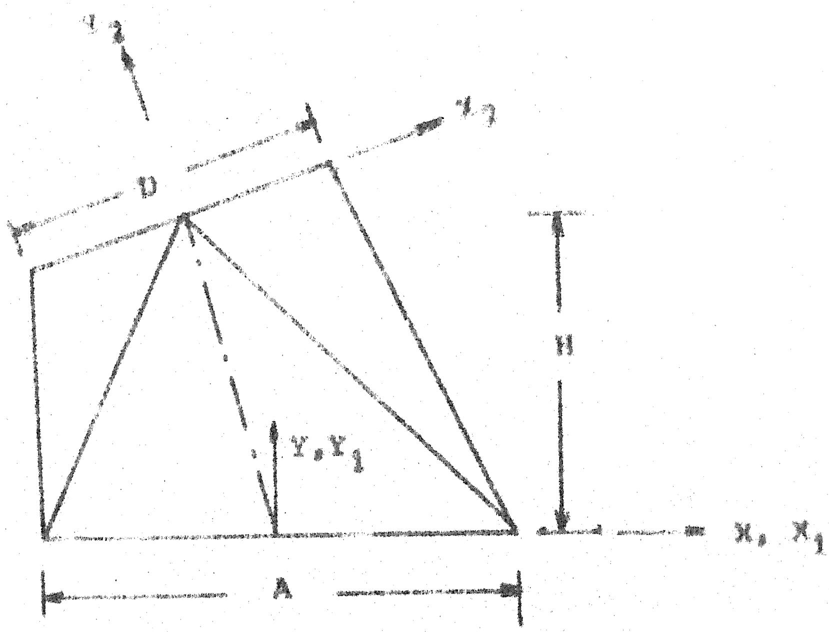
5.2.3 Example 3

As an example for the algorithm for finding the optimal rectangle to a given arbitrary polygon (described in Section 4.2) the developed surface of Example 2, which is a polygon, is considered. The input for this example is nothing but the development data given in Table 5.3. The output is in three forms.

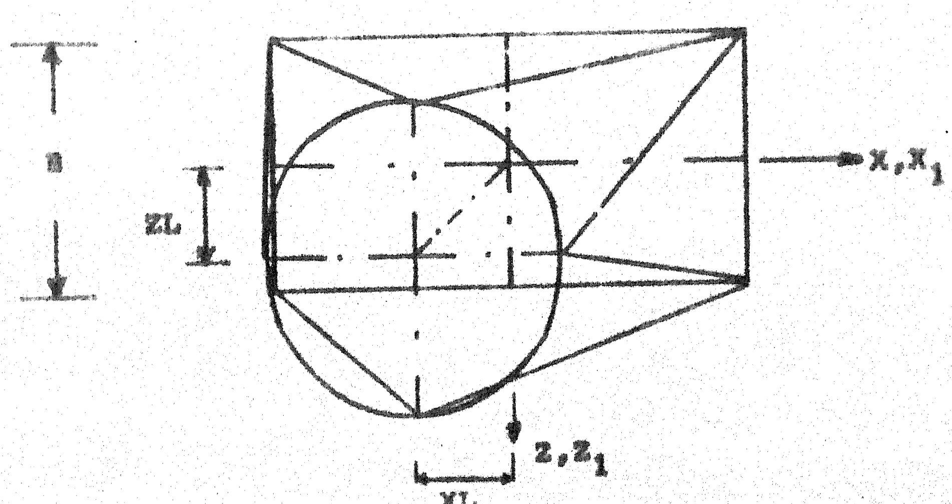
TABLE 5.2

Input data for Example 2

CURVE	CURVE NATURE	CURVE DEFINITION WITH RESPECT TO LOCAL COORDINATE SYSTEM. DIAMETER OR LENGTH AND WIDTH.	TRANSFORMATIONS				NUMBER OF PARTS THE PRIMARY CURVE DIVIDED INTO
			ROTA-TION ABOUT Z-AXIS	TRANSLATIONS			
				XL	YL	ZL	
PRIMARY	CIRCLE	10.0	20.0	-7.5	10.0	7.5	16
SECONDARY	RECTANGLE	15.0 X 8.0	0.0	0.0	0.0	0.0	



Front View



Top View

Fig. 5.4

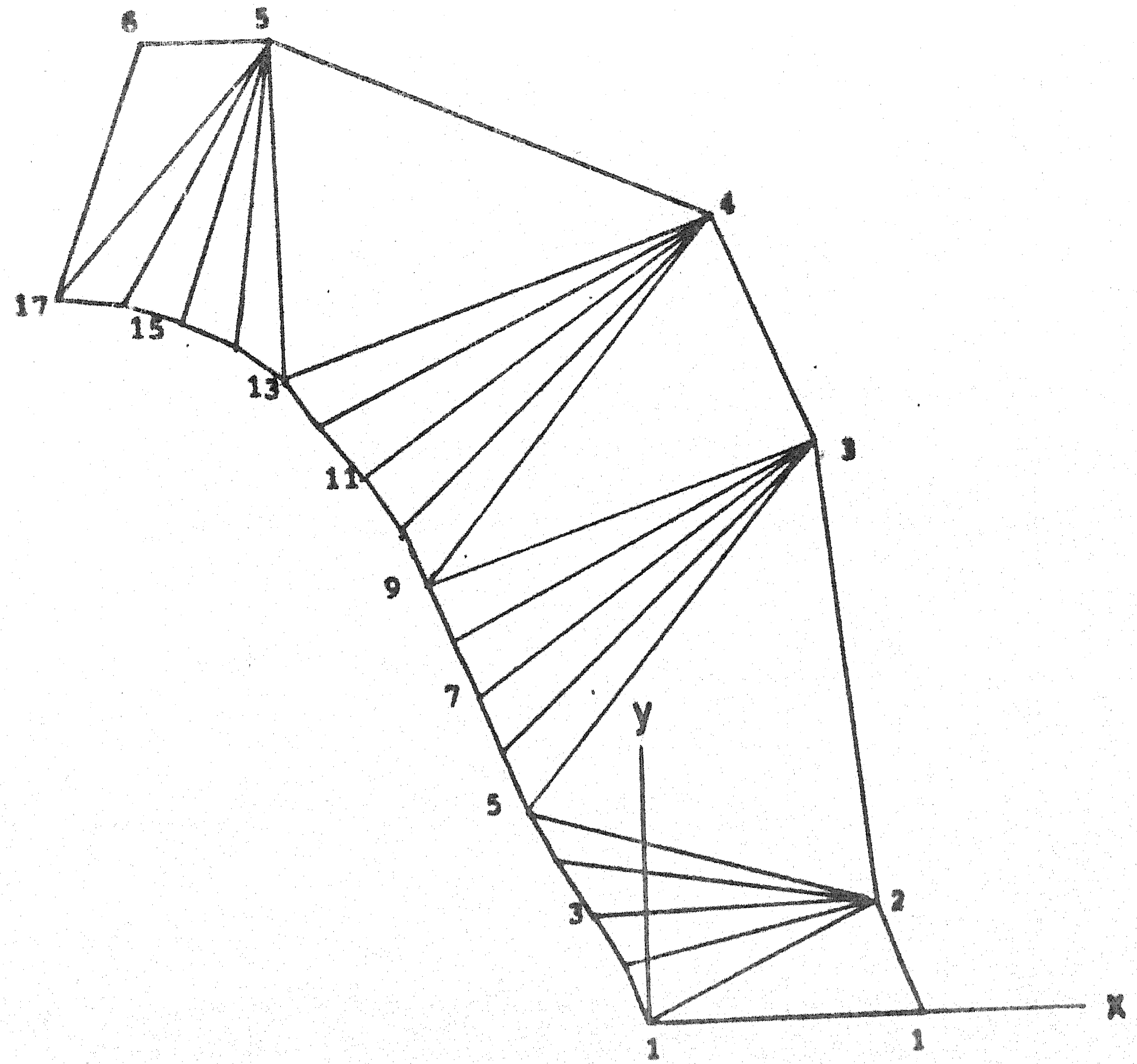


Fig. 5.5 Development

TABLE NO.5.3

POINTS	TOP	
	X	Y
1	0.0000	0.0000
2	-0.7834	1.8004
3	-1.7635	3.5018
4	-2.8289	5.1511
5	-3.7964	6.8597
6	-4.5800	8.6601
7	-5.2737	10.4969
8	-5.9784	12.3296
9	-6.7519	14.1344
10	-7.6256	15.8927
11	-8.6746	17.5525
12	-9.9230	19.0681
13	-11.3307	20.4369
14	-12.8828	21.6396
15	-14.6302	22.5349
16	-16.5291	23.0346
17	-18.4891	23.1522

POINTS	BASE	
	X	Y
1	8.8183	0.0000
2	7.4575	3.7614
3	5.5559	18.6404
4	2.2403	25.9210
5	-11.6516	31.5790
6	-15.6509	31.5012

(i) The given arbitrary polygon with the encasing optimal (minimum-area) rectangle obtained in the form of plotter output is shown in Figure 5.6.

(ii) The x-y coordinates of different points of the polygon with respect to the corner coordinate system S_c (O_c - X_c Y_c) (shown in Figure 5.6) is obtained in the form of line-printer output and is given in Table 5.4.

(iii) The area of the developed surface, computed by using trapezoidal rule, the area of the optimal encasing rectangle, and the percentages of used and unused sheet area are included in Table 5.4.

5.3 Illustrative Examples

A large variety of surfaces were modelled and their development views were obtained using the four programs developed in the present work. Out of these several cases two cases have been included as the illustrative examples.

Figure 5.7 shows the front view and the top view of a frustum of right circular cone. The primary curve is a circle and the secondary curve is an ellipse. Note that the primary and the secondary curves are not

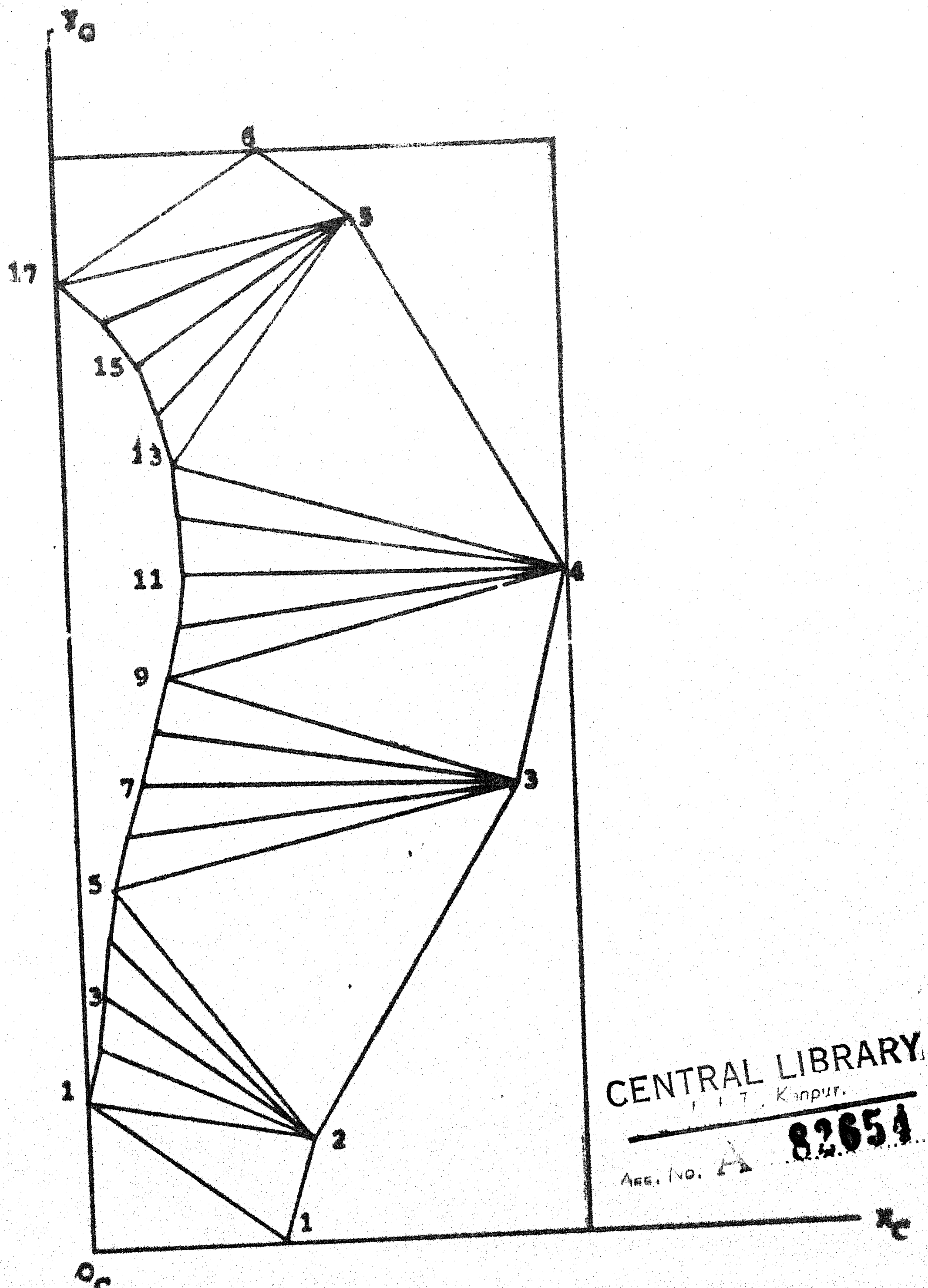


Fig. 5.6 Optimal (minimum area) encasing rectangle

TABLE 5.4

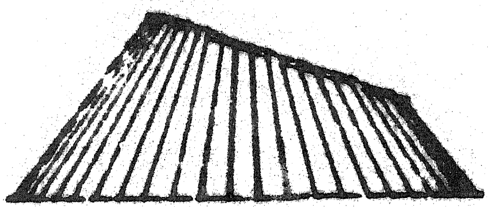
POINTS	TOP X	TOP Y
1	-0.000	5.5028
2	0.5114	7.3985
3	0.8072	9.3396
4	1.0039	11.2932
5	1.3140	13.2321
6	1.8253	15.1278
7	2.4294	16.9961
8	3.0224	18.8679
9	3.5442	20.7608
10	3.9587	22.6800
11	4.1747	24.6316
12	4.1450	26.5949
13	3.8991	28.5429
14	3.4369	30.4512
15	2.6301	32.2413
16	1.4582	33.8167
17	-0.0000	35.1316

POINTS	BASE X	BASE Y
1	6.8907	0.0000
2	8.1745	3.7884
3	15.9734	16.6015
4	17.9259	24.3596
5	10.6013	37.4497
6	7.4277	39.8846

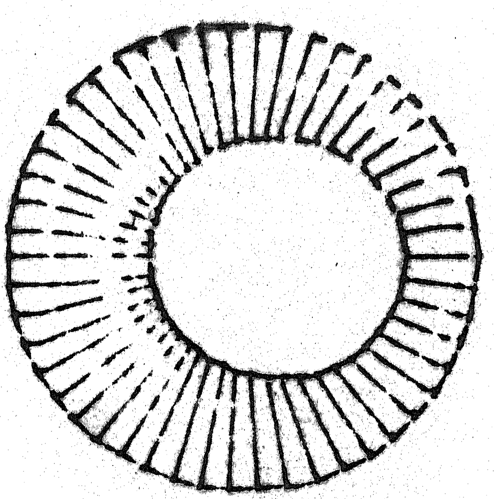
Area of the developed surface = 414.99326 sq. units
Area of the optimal circumscribing rectangle = 714.96602 sq. units
Percentage of the optimal rectangular sheet used = 58.0438
And the percentage of waste = 41.9562

parallel to one another. The development view of this frustum is shown in Figure 5.8. One can easily verify that all the generators of the cone pass through a common point in the development view. The Figure shown in 5.8 has been obtained as an output of the analytical method.

Figure 5.9 shows the elevation and the plan of a transition piece which is used as a spout. The primary curve is a circle and the secondary curve is a sixteen sided polygon. Using the modified triangulation method the development view obtained is also included in Figure 5.9.



Front view



Top view

Fig. 5.7

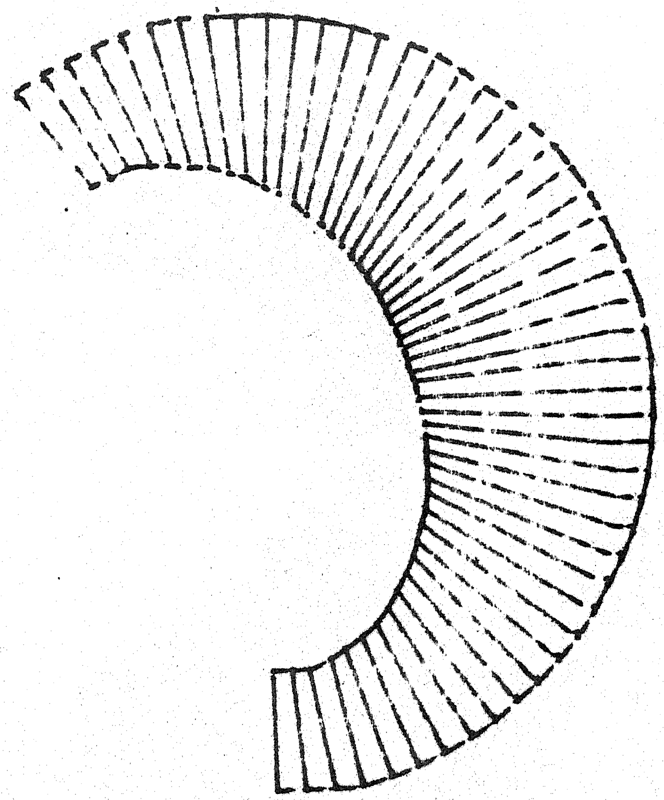


Fig. 5.8 Development

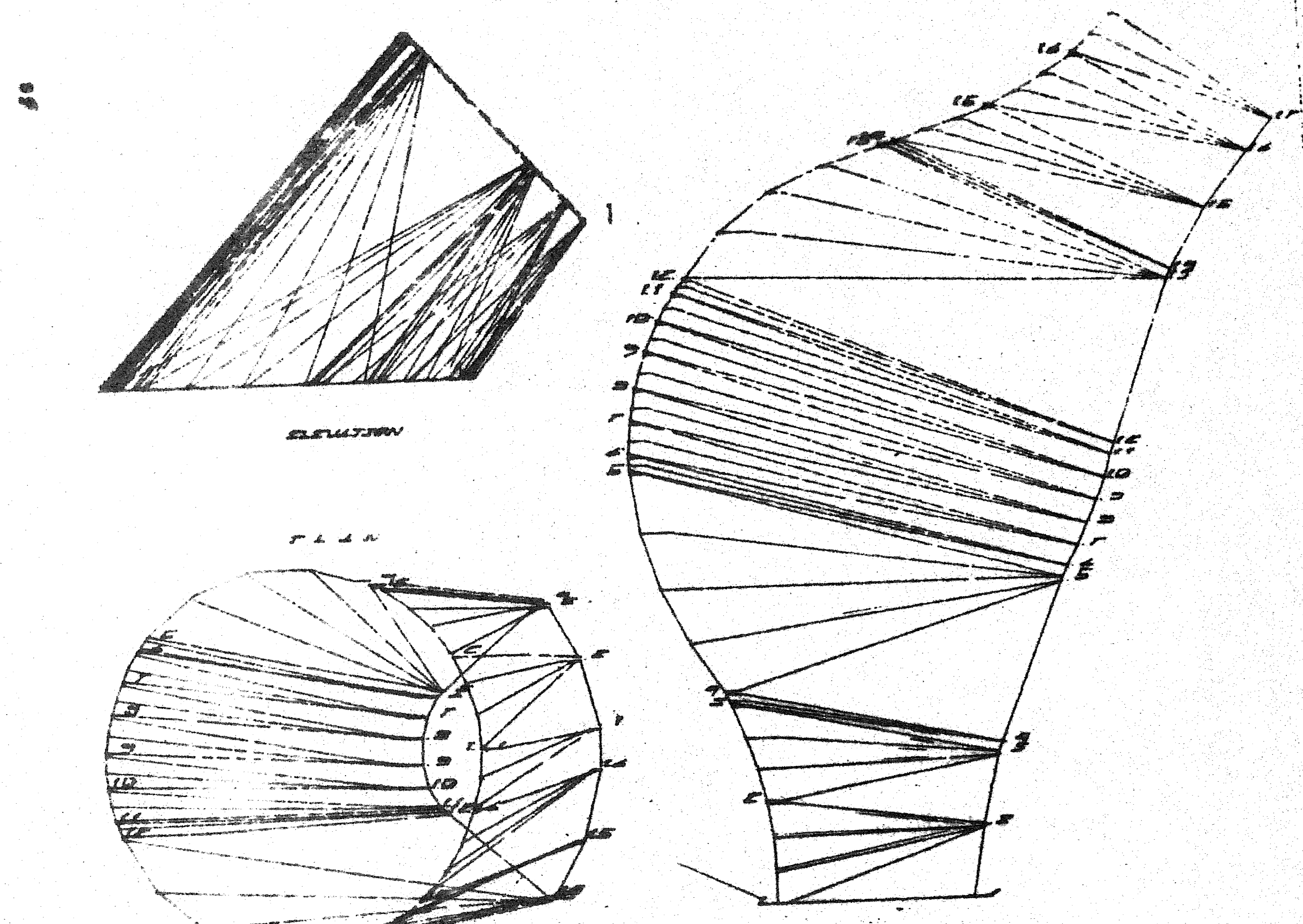


Fig. 5.9 Development of spout

CHAPTER-6

CONCLUSIONS

6.1 Technical Summary

Designers and draftsmen use many graphical techniques based on descriptive geometry to present the information about engineering components or assemblies. It is often necessary to get a development in a two dimensional plane of three dimensional surface, if it is developable. The problem is especially important while designing ducts, turbine casings, hoppers and spouts. The prevalent graphical techniques are based on the method of triangulation. This method is to some extent approximate in nature and it has limited applicability.

The emerging technology of computer graphics offers a powerful tool to designers enabling them to solve many of their problems of design and graphics. In the present work an attempt is made to show how the development of surfaces can be obtained on a direct-view storage tube (DVST) graphics terminal connected to computer. The method is based on the principle that the geodesic curvature of a curve lying on a developable surface is the same as the curvature of the corresponding

curve on the developed surface lying on the tangent plane.

Based on the above mentioned principle an algorithm has been developed and tested using several illustrative examples. The salient features of the analytical as well as the computational work can be listed as follows:

(i) The program developed based on the analytical method given in Chapter 2, is sufficiently general in nature. It is possible to develop any surface which satisfies the basic condition that it is developable. Transition pieces are composite surfaces consisting of several piece-wise developable surfaces. These can also be developed using the analytical method.

(ii) The analytical method is computationally not so efficient for those surfaces which are simple or degenerate in nature. In such cases the method of triangulation seems to be a better choice. In the present work, a modified method of triangulation has been developed.

(iii) The problem of finding an encasing rectangle to the developed view of a surface is as

important as getting the developed view itself. It is necessary that the encasing rectangle be such that the unused or the waste area is minimum. In the present work an algorithm has been developed to find the optimal (minimum area) encasing rectangle to a given polygon. Once the optimal encasing rectangle is obtained, the algorithm is also used to compute the coordinates of all vertices of the polygon with respect to the corner-coordinate system.

(iv) All the programs developed have a graphics interface, the graphics package used is the GPGS (General Purpose Graphics Systems), and the graphics hardware used is the Tektronics 4006 terminal and the Tektronics 4662 plotter.

(v) All the programs have been tested successfully on a variety of problems and have been found to be computationally efficient.

6.2 Recommendations for Further Work

(i) The present work deals with those surfaces which satisfy the condition of developability. In practice sometimes the surfaces are warped, or double curved. Such surfaces though strictly

not developable, are still formed out of a developed plane sheet. It is necessary to take into account such constraint and modify or develop an algorithm for getting the development of a warped surface. The tipical examples of such cases are the developed view of shoes and dresses.

(ii) Besides development, one often needs to get the intersection of two surfaces and to get the development of resulting surfaces. The present work can be extended to develop algorithms based on computational geometry to get the intersection of developable surfaces.

(iii) The present version of the programs depend on the GPGS graphics software and the PASCAL and FORTRAN compilers. It is desirable to develop a version of the package which is device independent and which uses commonly available software.

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APPENDIX-I

In this section, fourth order Runge-Kutta method for the integration of Serret-Frenet equations (2.10) is explained. The equations are

$$\frac{d^2x}{ds^2} + k(s) \frac{dy}{ds} = 0$$
$$\frac{d^2y}{ds^2} - k(s) \frac{dx}{ds} = 0 \qquad (2.10)$$

Let $y_1 = x$ (I.1)

$$y_2 = \dot{x} = \frac{dx}{ds} \qquad (I.2)$$

$$y_3 = y \qquad (I.3)$$

$$y_4 = \dot{y} = \frac{dy}{ds} \qquad (I.4)$$

Now from Equations (I.1) and (I.2)

$$\frac{dy_1}{ds} = y_2 \qquad (I.5)$$

From (I.2), (2.10) and (I.4)

$$\frac{dy_2}{ds} = - k(s) \; y_4 \qquad (I.6)$$

From (I.3) and (I.4)

$$\frac{dy_3}{ds} = y_4 \qquad (I.7)$$

From (I.4),(2.10)and (I.2)

$$\frac{dy_4}{ds} = k(s)\ y_2 \tag{I.8}$$

The Equations I.5 to I.8 form a system of first order ordinary differential equations and the integration can be done using the standard numerical fourth order Runge-Kutta method [5].

APPENDIX-II

In this section, a method to find out if the line defined by any two vertices of an n-vertex polygon trans-passes over the area of the polygon is described.

Let a polygon be defined by its n vertices $V_1, V_2, \ldots, V_n$. Satisfy the requirements 4.2.1 and 4.2.2.

As shown in Figure II.1 let $\underline{r}_i$, $\underline{r}_{i+j}$ and $\underline{r}_k$ be any three vectors defined by three vertices V_i, V_{i+j} and V_k respectively of the polygon. It can be seen that

$$CP \equiv (\underline{r}_{i+j} - \underline{r}_i) \times (\underline{r}_k - \underline{r}_{i+j}) \gtreqless 0 \quad \text{(II.1)}$$

Case 1 : CP>0. This means that the vertex V_k is to the left of the vector $(\underline{r}_{i+j} - \underline{r}_i)$

Case 2 : CP=0. The vertex V_k is on the vector $(\underline{r}_{i+j} - \underline{r}_i)$ or its extended portion.

Case 3 : CP<0. The vertex V_k is to right of the vector $(\underline{r}_{i+j} - \underline{r}_i)$.

Now given V_i, one can find V_{i+j} such that all the remaining vertices satisfy the condition CP > = 0. If this condition is satisfied then the vertices V_i and V_{i+j} define an encasing rectangle.

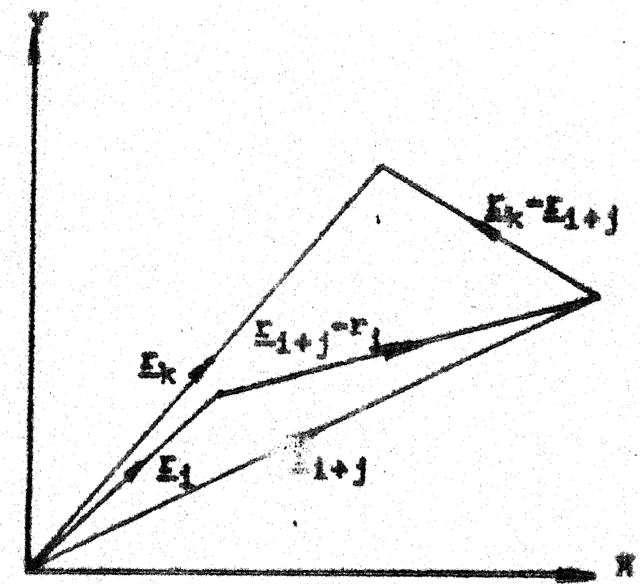
Y
$\underline{r}_k - \underline{r}_{i+j}$
$\underline{r}_{i+j} - r_i$
$\underline{r}_k$
$\underline{r}_i$
$\underline{r}_{i+j}$
X

Fig. II 1

APPENDIX-III

In this section, a method to findout the corners of an encasing rectangle for a given arbitrary polygon is explained.

Step 1 : Let the polygon be defined by its n vertices V_1, $V_2, \ldots V_n$. Also let V_i and V_{i+j} be the pair of vertices which define the encasing rectangle, that is, the line formed by joining and extending V_i and V_{i+j} does not transpass over the area of the polygon. $\underline{r}_i$ and $\underline{r}_{i+j}$ are the corresponding vectors. Let $\underline{r}_k$ be another radius vector defined by any other vertex V_k.

Step 2 : Find $D_k \equiv (\underline{r}_{i+j} - \underline{r}_i) \cdot (\underline{r}_k - \underline{r}_{i+j})$ for all the vertices V_k, $k = i + j + 1$ to n and $k = 1$ to $i - 1$.

Step 3: Find the value of f for which D_f is maximum. If $D_f < 0$ then V_{i+j} itself is a corner of the rectangle. Otherwise, it is the point of intersection of the line through V_i and V_{i+j} and the line through V_f and perpendicular to the previous line (through V_i and V_{i+j}).

In a similar way other corners of the encasing rectangle can be found.